# Section 12.4:
# The Cross Product

# Things we'll go over today…

Section 12.4: The Cross Product

- 2x2 and 3x3 determinants
- Definition of the Cross Product
- Some results on the Cross Product
- Geometric interpretation of the Cross Product
- Properties of the Cross Product
- Triple Products
- Torque

# 1. 2x2 and 3x3 Determinants

Given a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its determinant is the number given by…

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

Ex 1: Find $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$

# 1. 2x2 and 3x3 Determinants

Explain 3x3 determinants…

Ex 2: Find $\begin{vmatrix} -1 & 4 & 0 \\ 2 & 2 & 3 \\ 4 & 1 & -5 \end{vmatrix}$ (twice)

# 2. Definition of the Cross Product

Given 2 vectors $\vec{a}$ & $\vec{b}$, the goal is to come up with a vector perpendicular to both. One way of doing this is with the cross product…

Def: The cross product of 2 vectors $\vec{a} = < a_1, a_2, a_3 >$ & $\vec{b} = < b_1, b_2, b_3 >$ is the vector …

$$\vec{a} \times \vec{b} \equiv \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where you must expand along the $1^{st}$ row.

Note: The cross product is only defined for vectors in $V_3$

# 2. Definition of the Cross Product

Ex 3: Find the cross product $\vec{v} \times \vec{w}$ of the vectors

$\vec{v} =< 5, 2, -1 >$ & $\vec{w} =< 4, -3, -7 >$

Ex 4: Find the cross product $\vec{a} \times \vec{b}$ of the vectors

$\vec{a} = 2\boldsymbol{i} - 6\boldsymbol{j}$ & $\vec{b} = 3\boldsymbol{j} + \boldsymbol{k}$

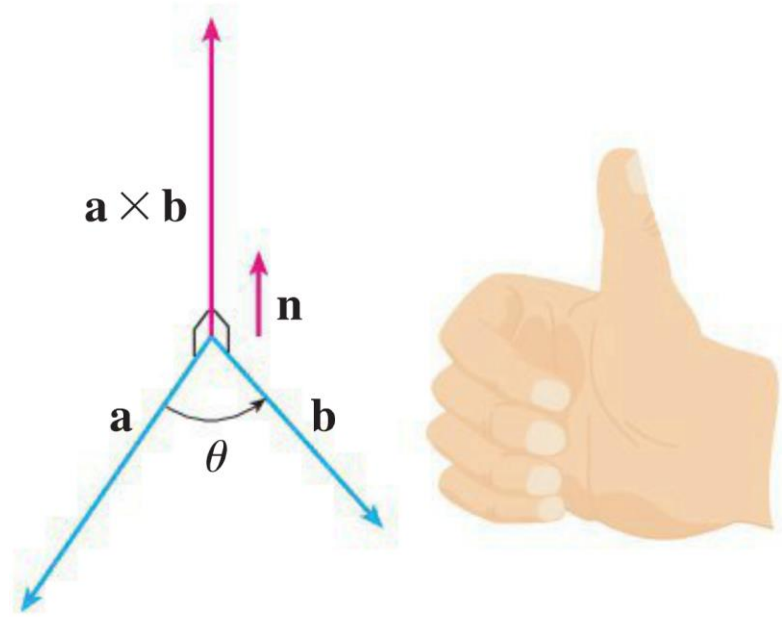
# 3. Some Results On The Cross Product

**8 Theorem** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof: …

More specifically…

Right Hand Rule



# 3. Some Results On The Cross Product

**9 Theorem** If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Proof: …

**10 Corollary** Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

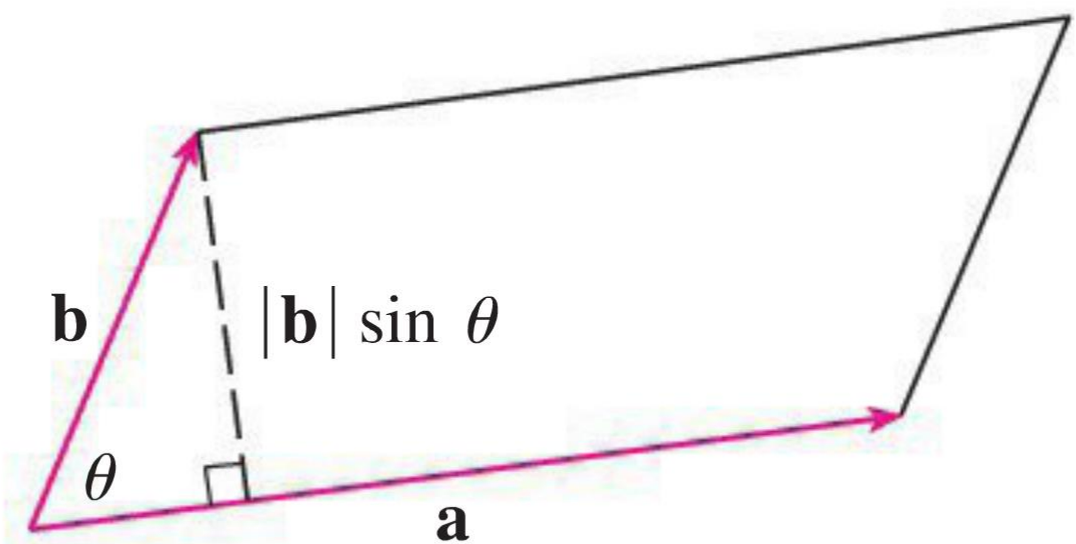
$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Proof: …

# 4. Geometric Interpretation of the Cross Product

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Proof: …



# 4. Geometric Interpretation of the Cross Product

Ex 5: Find a vector that is perpendicular to the plane that passes through the points $P(3,2,0)$ , $Q(-1,5,2)$ , & $R(-2,1,-2)$

# 4. Geometric Interpretation of the Cross Product

Ex 6: Find the area of the triangle with vertices $P(3, 2, 0)$, $Q(-1, 5, 2)$, & $R(-2, 1, -2)$

# 5. Properties of the Cross Product

| | | |
|---|---|---|
| $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ | $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ | $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ |
| $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ | $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ | $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ |

Prove some of these…

# 5. Properties of the Cross Product

**11 Properties of the Cross Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and $c$ is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Prove some of these…

# 6. Triple Products

Def: If $\vec{a}$ , $\vec{b}$ , & $\vec{c}$ are vectors in $V_3$ , the scalar

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$$

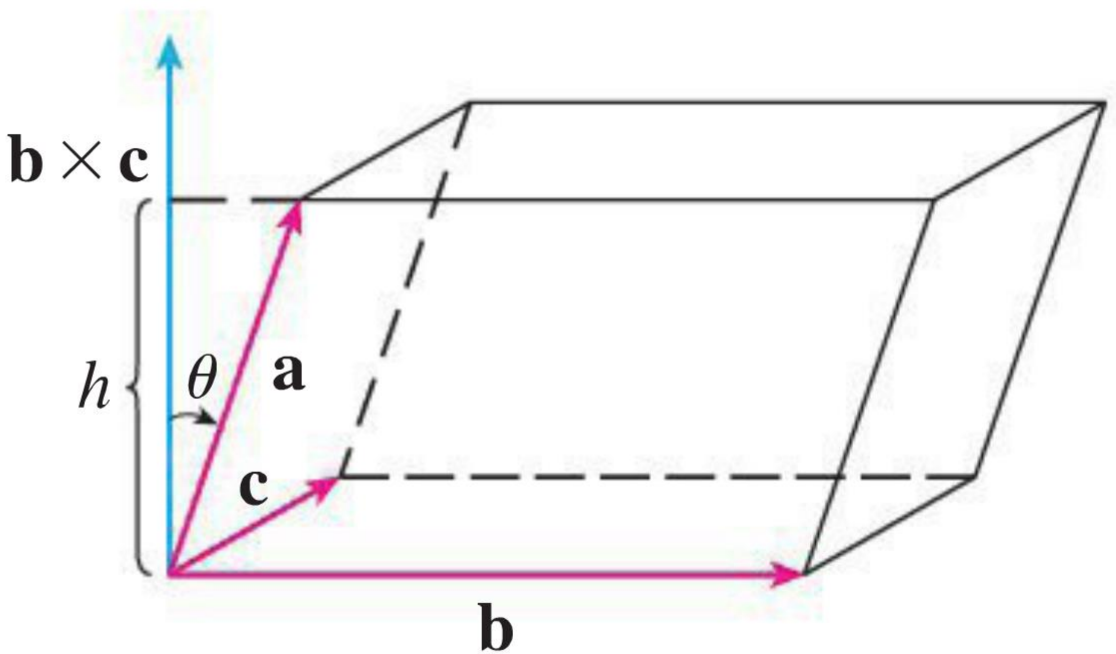
is called the scalar triple product.

Formula: If $\vec{a} =< a_1, a_2, a_3 >$ , $\vec{b} =< b_1, b_2, b_3 >$, & $\vec{c} =< c_1, c_2, c_3 >$ , …

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

# 6. Triple Products

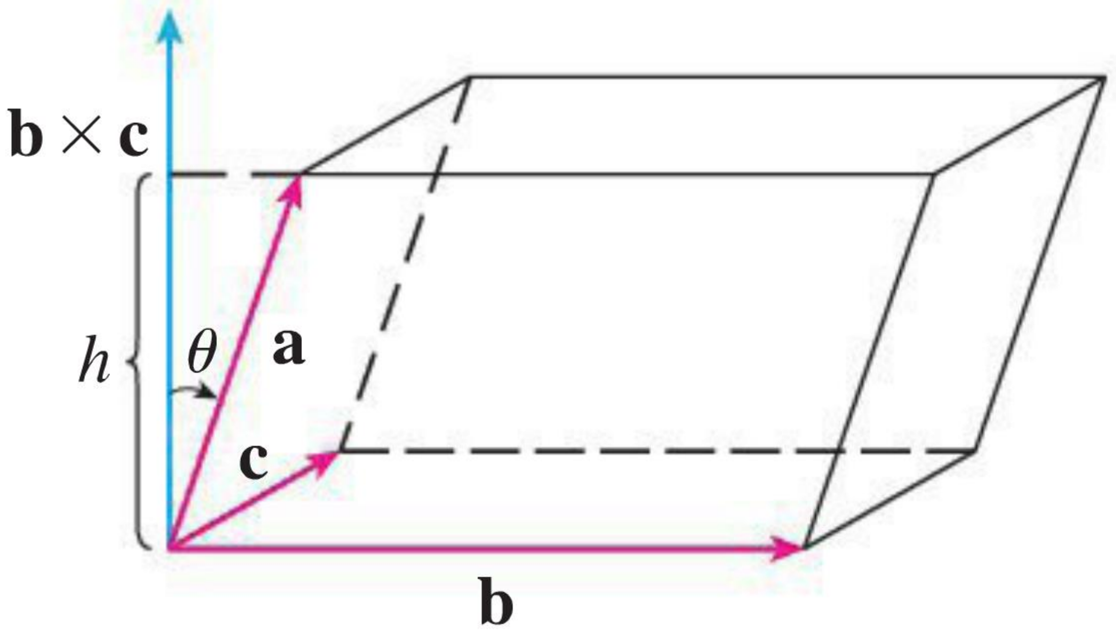
<u>Geometric Interpretation</u>: $\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$ is the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, & $\vec{c}$



# 6. Triple Products

**14** The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



# 6. Triple Products

Ex 7: Find the volume of the parallelepiped determined by the vectors $\vec{a} = < 2, 5, -1 >$, $\vec{b} = < 1, 1, 1 >$, & $\vec{c} = < 4, -2, 1 >$

# 6. Triple Products

Ex 8: Show that the vectors $\vec{a} =< 1, 4, -7 >$, $\vec{b} =< 2, -1, 4 >$, & $\vec{c} =< 0, -9, 18 >$ are coplanar.

# 6. Triple Products

<u>Def</u>: If $\vec{a}$ , $\vec{b}$ , & $\vec{c}$ are vectors in $V_3$ , the vector

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right)$$

is called the <u>vector triple product</u>.

# 7. Torque

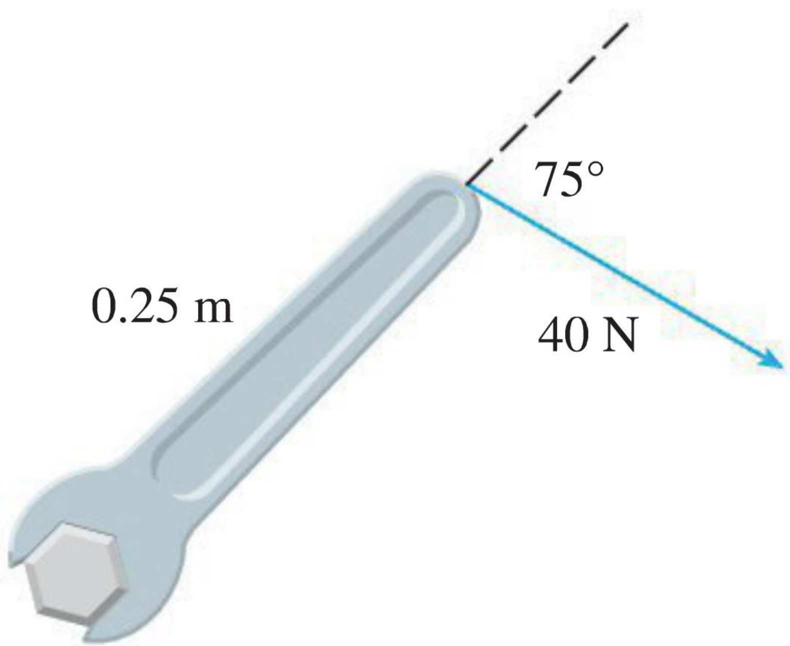
Def: Suppose a rigid body is free to rotate about a point (called the pivot). If $\vec{F}$ is a force applied to another point P on the rigid body in an effort to make the rigid body rotate, and if $\vec{r}$ is the vector whose tail is the pivot point and whose head is the point P where the force is being applies, then the torque applied to the rigid body about the pivot is the vector $\vec{\tau}$ defined by…

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

# 7. Torque

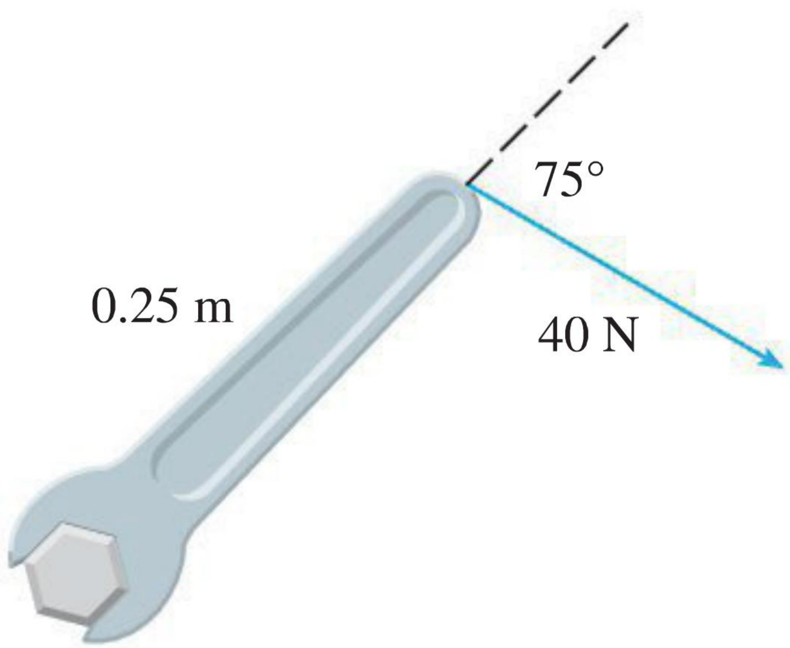
Def: $$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

Note: You can think of torque as a force of rotation (even though torque is not a force)



# 7. Torque

Ex 9: A bolt is tightened by applying a 40 N force to a 0.25 m wrench as in the picture below. Find the magnitude of the torque about the center of the bolt.



# 7. Torque

Ex 10: A bicycle pedal is being pushed by a 60 N force as shown in the picture below. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about the point P.